2013

Research Question: How do you find the area under a curve numerically?

Topic:

Numerical methods of finding area under the curve

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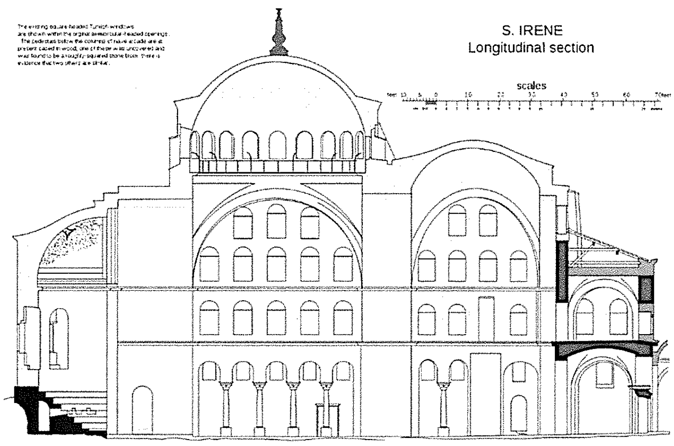
Introduction

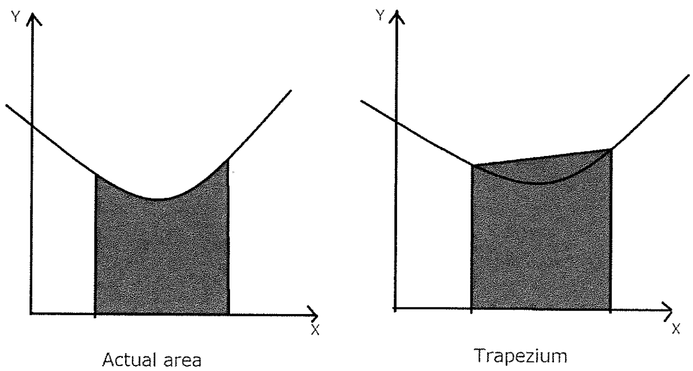


In this essay, we will find and use two different ways of numerical methods which are trapezium rule and Simpson's rule to find area under the curve. I am interested in architecture and I wondered how do we find the area or volume of complex object, and the interests made me research this topic. In addition, analytical method is used in our mathematics standard level lesson, which also make me interested in this topic.

The numerical method stands for "the study which relates the technique to find approximately method on problems which is impossible to find the answers algebraically". Area under the curve can be sometimes algebra because sometimes the graph is curve and it is hard to find the exact area under the curve. In real life, numerical method is able to use in architecture. For example, Byzantine Architecture can be used by the numerical method. Byzantine Architecture is one of the architecture style which was rose under the forces of Eastern Europe Empire in 4th century. As upper cross section diagram of building which is built to use the Byzantine Architecture, one of the characteristic is that there are lots of dome shape in this building style. If we want to find the area or volume of the building, and if we use only usual simple equations, it is really hard to find them. Because there are not only rectangles and half circle, but also little bit complicated shapes on the building. Therefore, the numerical method of finding area under the curve is going to be useful and practical. If we use it to find the area or volume of the building, we only have to know equations of curve $(f(x))$ and we easily find it. This solution going to be used not only to find the area or volume of buildings, but also find area of balcony, staircase, loft, attic and pond. From those explanations, the numerical method of finding area under the curve is used variously especially in architecture. There are three ways to find them. Trapezium rule, Simpson's rule and definite integration. This EE introduce about how to find and use them, and the difference between those three equations. So that is how I will answer my research question.

The Trapezium rule

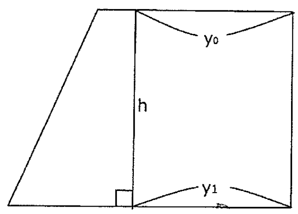
It is one of the function ~~to~~ find ~~the~~ approximating the area under a curve to use the trapezoid.



There are three process to find the area under the curve to use the trapezium rule.

1. Find trapezium and measure how much is the upper line, downer line and height of the trapezium.
2. Find each areas of the trapeziums.
3. Add all of them up.

Firstly, we should remind the how to find the area of the trapezium.



From that shape, we can find the area of the trapezium following equation:

$$\text{Area of trampezium} = h\left(\frac{y_0 + y_1}{2}\right) \ldots ①$$

Here is how the trapezium rule works when there are three adjoining trapezia. As following the diagram, there are three different kinds of trapeziums. If we add them up,

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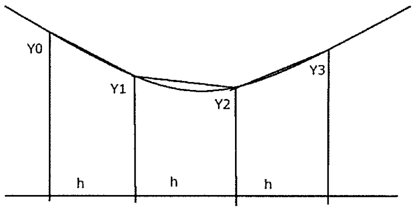
we can find approximate area under the curve. Y_0, Y_1, Y_2, Y_3 are all different height and the distance between their parallel sides (h) is all same length.



If we use formula①, area under the curve can calculated that

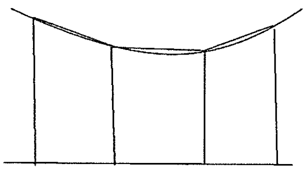
$$\left(\frac{y_0+y_1}{2}\right)h+\left(\frac{y_1+y_2}{2}\right)h+\left(\frac{y_2+y_3}{2}\right)h$$

$$=\frac{h}{2}(y_0+2y_1+2y_2+y_3)$$

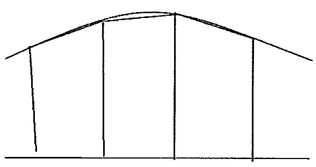
From this example, we can find this formula (if total of the area under the curve shows T),

$$\boldsymbol{T=\frac{h}{2}(y_0+2y_1+2y_2+\cdots+2y_{n-1}+y_n)}$$

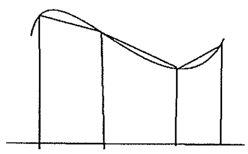
We can separate them smaller trapeziums. If we do so we can find more accurate area. The answer of trapezium rule approximation can be greater than or less than actual area (A). It is depending on whether the curve of f(x) is concave or convex. It easily judge the area is greater or smaller than actual area.



convex (T > A)



concave (T < A)



Neither (T ≈ A)

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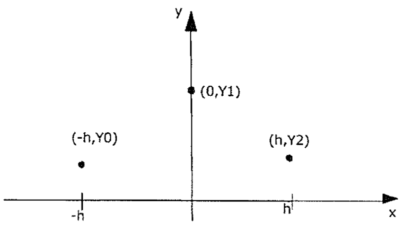
Simpson's Rule

It basically linear approximates a curve with series of line segment. It tends to remove errors and gives a very accurate estimate of the total area under the curve.

I am going to explain how to find Simpson's rule. We will use the areas under a set of parabolic arcs.

1. Consider the three points spaced about the y-axis. The general equation of parabola through these three points is $y = ax^2 + bx + c$.



And the area A is given by the definite integral between the limits h and $-h$.

$$A = \int_{-h}^{h} (ax^2 + bx + c)\,dx$$

$$= \left[\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right]_{-h}^{h}$$

$$= \left(\frac{ah^3}{3} + \frac{bh^2}{2} + ch\right) - \left(\frac{a(-h)^3}{3} + \frac{b(-h)^2}{2} - ch\right)$$

$$= \frac{2ah^3}{3} + 2ch$$

This area is express in terms of a and c.

2. Next, we will work out a and c using that the points lie on the curve.

If $(0,y_1)$ lies on the curve $y = ax^2 + bx + c, y_1 = 0 + c$, therefore $y_1 = c$.

So the curve becomes $y = ax^2 + bx + y_1$.

If $(-h, y_0)$ and (h, y_2) lie on the curve:

$$y_0 = ah^2 - bh + y_1$$
$$y_2 = ah^2 + bh + y_1$$

Then,

$$y_0 + y_2 = 2ah^2 + 2y_1$$
$$2ah^2 = y_0 - 2y_1 + y_2$$

From the result of part1, we ~~can find that~~

$$A = \frac{2ah^3}{3} + 2ch$$

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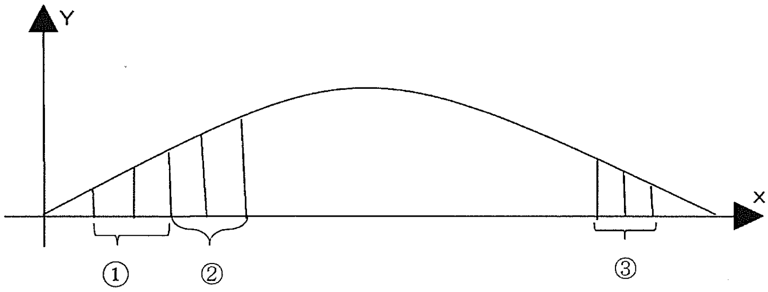
$$= \frac{2ah^2 \times h}{3} + 2ch$$

$$= \frac{(y_0 - 2y_1 + y_2)h}{3} + 2y_1 h$$

$$= \frac{h}{3}[y_0 - 2y_1 + y_2 + 6y_1]$$

$$= \frac{h}{3}[y_0 + 4y_1 + y_2]$$

3. Let's consider the whole curve with a series of several parabolic arcs:



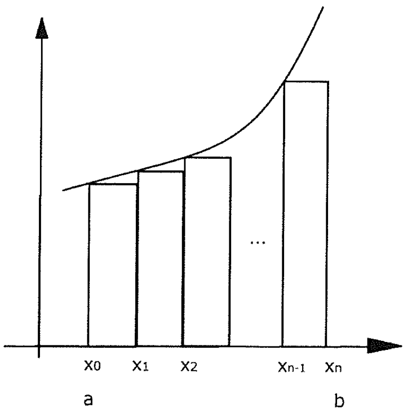
□ $A = \frac{h}{3}[y_0 + 4y_1 + y_2]$ ② $A = \frac{h}{3}[y_2 + 4y_3 + y_4]$ ③ $A = \frac{h}{3}[y_{n-2} + 4y_{n-1} + y_n]$

And we can find the formula of the Simpson's rule:

$$\textbf{Total Area} = \frac{h}{3}[y_0 + y_n + 4(y_1 + y_3 + \cdots y_{n-1}) + 2(y_2 + y_4 + \cdots y_{n-2})]$$

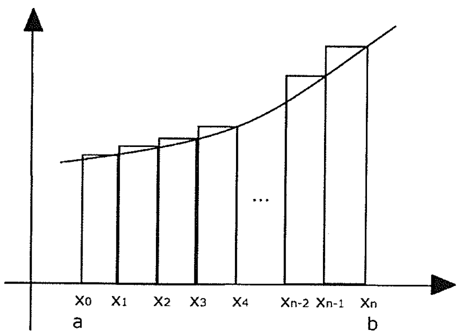
Definite Integration

If we use it to find the area under the curve, we can find most exact area. To find the rule of it, the lower and upper rectangles can use to help it.



Lower rectangles... If we want to find the total of all rectangles of a to b, the equation is going to be;

$$\text{Area of Lower rectangles} = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2) + \cdots + f(x_{n-2})\Delta x + f(x_{n-1})\Delta x$$
$$= \sum_{i=0}^{n-1} f(x_i)\Delta x \text{ where } \Delta x = \frac{b-a}{n}$$



Upper rectangles... if we want to find the total of all rectangles of a to b, he equation is going to be;

$$\text{Area of upper rectangles} = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3) + \cdots + f(x_{n-1})\Delta x + f(x_n)\Delta x$$
$$= \sum_{i=1}^{n} f(x_i)\Delta x \text{ where } \Delta x = \frac{b-a}{n}$$

We can define the number of the area under the curve which is between all lower sums

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and upper sums as $\int_a^b f(x)dx$. It is called “the definite integral of $f(x)$from a to b”.
i.e.

$$\sum_{i=0}^{n-1} f(x_i)\Delta x < \int_a^b f(x)dx < \sum_{i=1}^{n} f(x_i)\Delta x \text{ where } \Delta x = \frac{b-a}{n}$$

If n → ∞

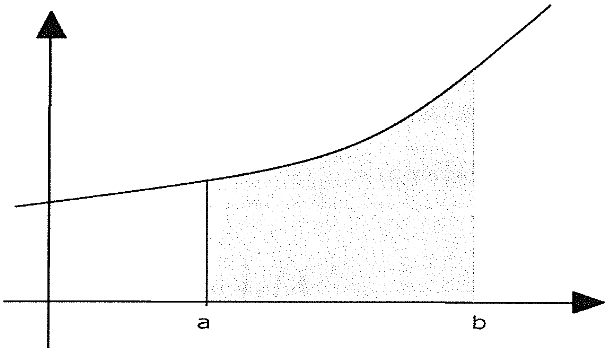
$$\sum_{i=0}^{n-1} f(x_i)\Delta x \rightarrow \int_a^b f(x)dx$$

And

$$\sum_{i=1}^{n} f(x_i)\Delta x \rightarrow \int_a^b f(x)dx$$

Therefore,

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx$$



From those equations, if we want to find the area under the curve (shaded area), which is all x on $[a, b]$ $(f(x) \geq 0)$, the area going to be $\int_a^b f(x)dx$
In addition, we can change the equation more easier way.
Assuming that

$$f(x) = F'(x)$$

and

$$\int_a^b f(x)dx$$

$$= [F(x) + c]_a^b$$

$$= F(b) + c - (F(a) + c)$$
$$= F(b) - F(a)$$

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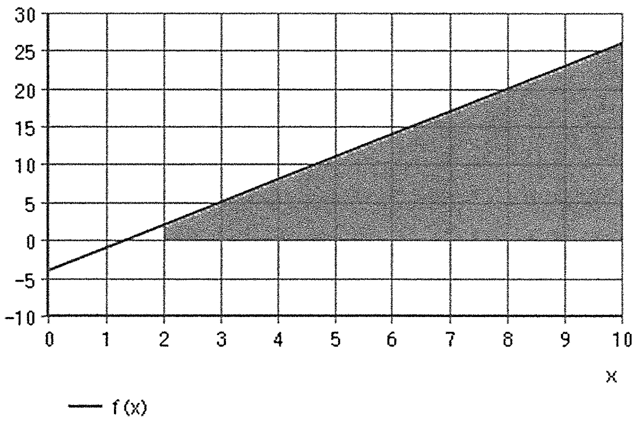
Therefore, the final equation is $\int_a^b f(x)dx = F(b) - F(a)$ **when** $f(x) = F'(x)$. It can be said that this equation is simplest and most accurate equation.

Example questions

1. $f(x) = 3x - 4 \dots (2 \le x \le 10)$



- **The Trapezium rule**

$$T = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n)$$

When we going to be solve this to use the trapezium rule, we should know the value of y of number of x. As I explained, if we separate number of x more, which means h become smaller, there will be more amount and smaller trapeziums, and the answer will be more accurate. In this question, I am going to find the value of y which is on each natural numbers of x.

So, $h = 1$

The number of section is $10 - 2 = 8$sections, so $n = 8$

And range of number of x is $(2 \le x \le 10)$, so points on $f(x)$ is going to be

$$(2, y_0), (3, y_1), (4, y_2), \dots, (9, y_7), (10, y_8)$$

And put on the equation, $f(x) = 3x - 4$

$$(2,2), (3,5), (4,8), \dots, (9,23), (10, 26)$$

Finally, put them on trapezium rule equation

$$T = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n)$$

$$= \frac{1}{2}(2 + 2 \times 5 + 2 \times 8 + 2 \times 11 + 2 \times 14 + 2 \times 17 + 2 \times 20 + 2 \times 23 + 26)$$

$$= \frac{1}{2}(2 + 10 + 16 + 22 + 28 + 34 + 40 + 46 + 26)$$

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$$=\frac{1}{2}\times 224 = \mathbf{112\ unit^2}$$

- **Simpson's Rule**

$$\boldsymbol{T = \frac{h}{3}[y_0 + y_n + 4(y_1 + y_3 + \cdots y_{n-1}) + 2(y_2 + y_4 + \cdots y_{n-2})]}$$

It uses approximates a curve with series of line segment. As same as trapezium rule, if the distance between x to another x becomes smaller, the answer becomes more accurate. And I will set same condition with the trapezium rule. So, $h = 1$

The numbers to use is same as trapezium rule as well, so I will use the position of points which I solved at trapezium rule section.

$$(2,2), (3,5), (4,8), \ldots, (9,23), (10,26)$$

Finally, I will put them on the Simpson's rule equation.

$$T = \frac{h}{3}[y_0 + y_n + 4(y_1 + y_3 + \cdots y_{n-1}) + 2(y_2 + y_4 + \cdots y_{n-2})]$$

$$=\frac{1}{3}[2 + 26 + 4(5 + 11 + 17 + 23) + 2(8 + 14 + 20)]$$

$$=\frac{1}{3}(2 + 26 + 4\times 56 + 2\times 42)$$

$$=\frac{1}{3}\times 336 \neq \mathbf{122\ unit^2}$$

- **Definite Integration**

$$\int_a^b \boldsymbol{f(x)dx} = \mathbf{F(b) - F(a)}\ \ \textbf{when}\ \ \boldsymbol{f}\mathbf{(x) = F'(x)}$$

On way of solving, I should integrate the equation.

The range of area is $(2 \le x \le 10)$, so $a = 2, b = 10$

If we put the $\mathrm{f}(x)$ to definite integration equation,

When $f(x) = \mathrm{F}'(\mathrm{x})$

$$\int_a^b f(x)dx = [F(x) + c]_a^b$$

$$= \int_2^{10} (3x - 4)\,dx = \left[\frac{3x^2}{2} - 4x + c\right]_2^{10}$$

$$= \left(\frac{3\times 10^2}{2} - 4\times 10 + c\right) - \left(\frac{3\times 2^2}{2} - 4\times 2 + c\right)$$

$$= (150 - 40 + \mathrm{c}) - (6 - 8 + \mathrm{c})$$

$$= 110 + 2 = \mathbf{112\ unit^2}$$

As we can see all answers, if the graph ($f(x)$) is straight, all answers are same in different types of solving. If we try to normal trapezium equation, (upper line = $y_0 = 2$, downer line = $y_8 = 26$, height = $2 \leq x \leq 10 = 8$)

$$\text{Total area} = h\left(\frac{y_0 + y_1}{2}\right)$$

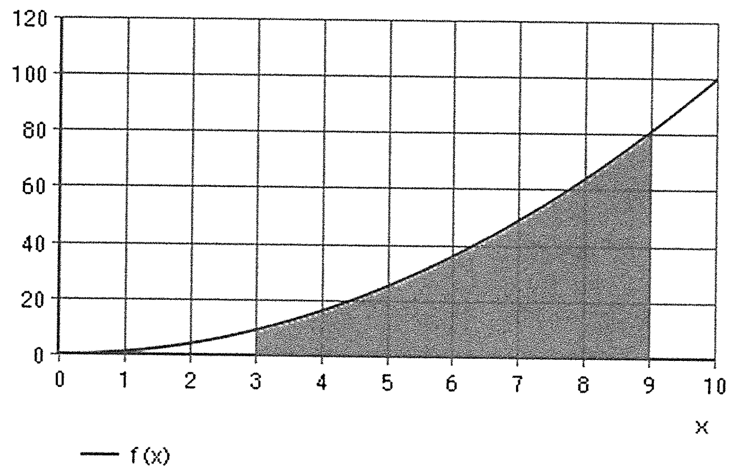
$$= 8\left(\frac{2 + 26}{2}\right) = \mathbf{112unit^2}$$

To measure accuracy of the area under the curve, to use graphic calculator can be the best way. I am going to use "TS-84 Plus" as a source.

To find the area enclosed by $f(x) = 3x - 4$, the x-axis, $x = 2$ and $x = 10$, we first draw the graph of $y = 3x - 4$. Press "2nd" "TRACE (CALC)" and select "7:Rf(x) dx". Press "2" "ENTER" "10" "ENTER" to specify the lower and upper limits of the integral.

The calculator shows us, area under the curve $f(x) = 3x - 4 \ldots (2 \leq x \leq 10)$ is 112 unit². Therefore, all the way of solving area under the curve are accurate to compare this calculator in this graph

2. $f(x) = x^2 \dots (3 \leq x \leq 9)$

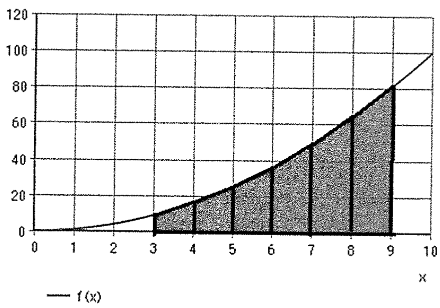


- **Trapezium rule**

  As same as last question, I am going to separate x to natural number.

  So, $h = 1$

  If $h = 1$, and if we try to solve this to use the trapezium rule, the shape going to be like this.



All objects are perfectly trapeziums.

The number of section is $9 - 3 = 6$sections, so $n = 6$

And range of number of x is $(3 \leq x \leq 9)$, so points on $f(x)$ is going to be

$$(3, y_0), (4, y_1), (5, y_2), \dots, (8, y_5), (9, y_6)$$

And put on the equation, $f(x) = x^2$

$$(3,9), (4,16), (5,25), \dots, (8,64), (9,81)$$

Finally, put them on trapezium rule equation

$$T = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n)$$

$$= \frac{1}{2}(9 + 2 \times 16 + 2 \times 25 + 2 \times 36 + 2 \times 49 + 2 \times 64 + 81)$$

$$= \frac{1}{2}(9 + 32 + 50 + 72 + 98 + 128 + 81)$$

$$= \frac{1}{2} \times 470 = \mathbf{235unit^2}$$

- **Simpson's rule**

  As same as last question, I am going to separate x to natural number.

  So, $h = 1$

  And we are going to use each points on the line, so we use the points that I solved at trapezium rule.

  $$(3,9), (4,16), (5,25), \dots, (8,64), (9,81)$$

  They can fit Simpson's rule equation

  $$T = \frac{h}{3}[y_0 + y_n + 4(y_1 + y_3 + \cdots y_{n-1}) + 2(y_2 + y_4 + \cdots y_{n-2})]$$

  $$= \frac{1}{3}[9 + 81 + 4(16 + 36 + 64) + 2(25 + 49)]$$

  $$= \frac{1}{3}(9 + 81 + 463 + 148)$$

  $$= \frac{1}{3} \times 701 = \mathbf{233.666 \dots unit^2}$$

- **Definite Integration**

  We are going to integrate this equation.

  The range of x is $(3 \le x \le 9)$. So, $a = 3, b = 9$

  $$\text{if } f(x) = F'(x)$$

  $$\int_a^b f(x)dx = [F(x) + c]_a^b$$

  $$= \int_3^9 x^2 dx = \left[\frac{x^3}{3} + c\right]_3^9$$

  $$= \left(\frac{9^3}{3} + c\right) - \left(\frac{3^3}{3} + c\right)$$

  $$= 243 - 9 = \mathbf{234\ unit^2}$$

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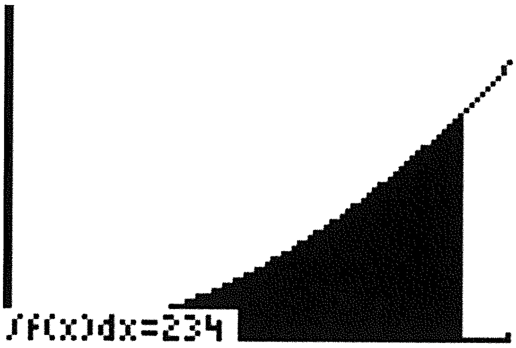
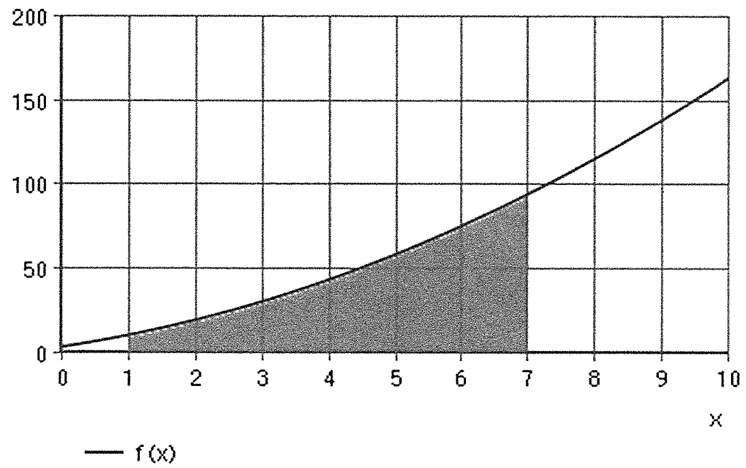
Let's see how accurate these equations in this graph. We are going to use same calculator and same way.



This calculator shows us area under the curve $f(x) = x^2$ ... $(3 \leq x \leq 9)$ is 234 unit^2. With comparing answers of trapezium rule and Simpson's rule with this calculator's answer, it is really similar, but not exactly same. However definite integration's answer is exactly same as the calculator's answer. Therefore, definite integration is most accurate solution in this graph.

3. $\boldsymbol{f(x) = x^2 + 6x + 3 \ldots (2 \le x \le 7)}$

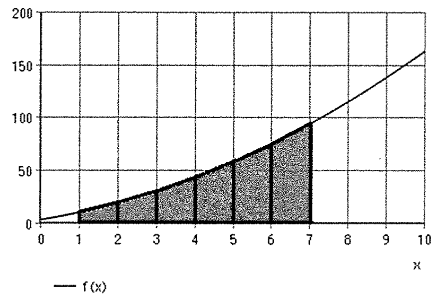


- **Trapezium rule**

  As same as last question, I am going to separate x to natural number.

  So, $h = 1$

  If $h = 1$, and if we try to solve this to use the trapezium rule, the shape going to be like this.



All objects are perfectly trapeziums.

The number of section is $7 - 1 = 6$sections, so $n = 6$

And range of number of x is $(1 \le x \le 7)$, so points on $f(x)$ is going to be

$$(1, y_0), (2, y_1), (3, y_2), \ldots, (6, y_5), (7, y_6)$$

And put on the equation, $f(x) = x^2 + 6x + 3$

$$(1,10), (2,19), (3,30), \ldots, (6,75), (7,94)$$

Finally, put them on trapezium rule equation

$$T = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n)$$

$$= \frac{1}{2}(10 + 2 \times 19 + 2 \times 30 + 2 \times 43 + 2 \times 58 + 2 \times 75 + 94)$$

$$= \frac{1}{2} \times 554 = \mathbf{277\ unit^2}$$

- **Simpson's rule**

  As same as last question, I am going to separate x to natural number.

  So, $h = 1$

  And we are going to use each points on the line, so we use the points that I solved at trapezium rule.

$$(1,10), (2,19), (3,30), \ldots, (6,75), (7,94)$$

  They can fit Simpson's rule equation

$$T = \frac{h}{3}[y_0 + y_n + 4(y_1 + y_3 + \cdots y_{n-1}) + 2(y_2 + y_4 + \cdots y_{n-2})]$$

$$= \frac{1}{3}[10 + 94 + 4(19 + 43 + 75) + 2(30 + 58)]$$

$$= \frac{1}{3} \times 828 = \mathbf{276\ unit^2}$$

- **Definite Integration**

  We are going to integrate this equation.

  The range of x is $(1 \leq x \leq 7)$. So, $a = 1, b = 7$

$$\text{if } f(x) = \mathrm{F}'(x)$$

$$\int_a^b f(x)dx = [F(x) + c]_a^b$$

$$= \int_1^7 (x^2 + 6x + 3)\,\mathrm{dx} = \left[\frac{x^3}{3} + \frac{6x^2}{2} + 3x\right]_1^7$$

$$= \left(\frac{343}{3} + \frac{294}{2} + 21\right) - \left(\frac{1}{3} + \frac{6}{2} + 3\right)$$

$$= \frac{847}{3} - \frac{19}{3} = \frac{828}{3} = \mathbf{276unit^2}$$

Let's see how accurate these equations in this graph. We are going to use same calculator and same way.

This calculator shows us area under the curve $f(x) = x^2 + 6x + 3 \ldots (2 \leq x \leq 7)$ is 276 unit². With comparing answers of trapezium rule and Simpson's rule with this calculator's answer, it is really similar, but not exactly same. However definite integration's answer is exactly same as the calculator's answer. Therefore, definite integration is most accurate solution in this graph.

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Conclusion

As the explanations and certification, there are three exactly different equations and there are some differences between the answers of equations. Trapezium rule is basically uses the trapeziums under the curve and find the approximate area. Simpson's rule uses linear approximates a curve with series of line segment and find the approximate area. The definite integration uses upper rectangles and lower rectangles of under the curve, and we can find that $\int_a^b f(x)dx$ is the equation of area under the curve. And it can defined that the equation is the most accurate and exact equation of area under the curve. Overall, the definite integration is most accurate and easiest equation in numerical method of area under the curve. However, sometimes it needs the calculator and sometimes uses algebra in the answer such as log or ln. Therefore, if we know the graph of $f(x)$, the trapezium rule and Simpson's rule are going to be useful to find the answer without the calculator or without using algebra.

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